Exercise 8

Use mathematical induction to verify the binomial formula (13) in Sec. 3. More precisely, note that the formula is true when $n = 1$. Then, assuming that it is valid when $n = m$ where m denotes any positive integer, show that it must hold when $n = m + 1$.

Suggestion: When $n = m + 1$, write

$$(z_1 + z_2)^{m+1} = (z_1 + z_2)(z_1 + z_2)^m = (z_2 + z_1)\sum_{k=0}^{m}\binom{m}{k} z_1^k z_2^{m-k}$$
$$= \sum_{k=0}^{m}\binom{m}{k} z_1^k z_2^{m+1-k} + \sum_{k=0}^{m}\binom{m}{k} z_1^{k+1} z_2^{m-k}$$

and replace k by $k - 1$ in the last sum here to obtain

$$(z_1 + z_2)^{m+1} = z_2^{m+1} + \sum_{k=1}^{m}\left[\binom{m}{k} + \binom{m}{k-1}\right] z_1^k z_2^{m+1-k} + z_1^{m+1}.$$

Finally, show how the right-hand side here becomes

$$z_2^{m+1} + \sum_{k=1}^{m}\binom{m+1}{k} z_1^k z_2^{m+1-k} + z_1^{m+1} = \sum_{k=0}^{m+1}\binom{m+1}{k} z_1^k z_2^{m+1-k}.$$

Solution

Use mathematical induction to prove the binomial theorem.

$$(z_1 + z_2)^n = \sum_{k=0}^{n}\binom{n}{k} z_1^k z_2^{n-k}, \quad n = 1, 2, \ldots \tag{13}$$

Check that the formula is satisfied for the base case $n = 1$.

$$(z_1 + z_2)^1 \stackrel{?}{=} \sum_{k=0}^{1}\binom{1}{k} z_1^k z_2^{1-k}$$

$$z_1 + z_2 \stackrel{?}{=} \sum_{k=0}^{1}\frac{1!}{k!(1-k)!} z_1^k z_2^{1-k}$$

$$\stackrel{?}{=} \frac{1!}{0!1!} z_1^0 z_2^1 + \frac{1!}{1!0!} z_1^1 z_2^0$$

$$\stackrel{?}{=} z_2 + z_1$$

$$= z_1 + z_2$$

The base case of the binomial formula holds.

Now assume the inductive hypothesis,

$$(z_1+z_2)^m = \sum_{k=0}^{m} \binom{m}{k} z_1^k z_2^{m-k}. \tag{1}$$

The aim is to show that

$$(z_1+z_2)^{m+1} = \sum_{k=0}^{m+1} \binom{m+1}{k} z_1^k z_2^{m+1-k}.$$

Start as suggested in the problem statement. Then use equation (1).

$$\begin{aligned}
(z_1+z_2)^{m+1} &= (z_1+z_2)(z_1+z_2)^m \\
&= (z_1+z_2)\sum_{k=0}^{m} \binom{m}{k} z_1^k z_2^{m-k} \\
&= z_1 \sum_{k=0}^{m} \binom{m}{k} z_1^k z_2^{m-k} + z_2 \sum_{k=0}^{m} \binom{m}{k} z_1^k z_2^{m-k} \\
&= \sum_{k=0}^{m} \binom{m}{k} z_1^{k+1} z_2^{m-k} + \sum_{k=0}^{m} \binom{m}{k} z_1^k z_2^{m+1-k}
\end{aligned}$$

Replace k with $k-1$ in the first sum.

$$\begin{aligned}
(z_1+z_2)^{m+1} &= \sum_{k-1=0}^{m} \binom{m}{k-1} z_1^{(k-1)+1} z_2^{m-(k-1)} + \sum_{k=0}^{m} \binom{m}{k} z_1^k z_2^{m+1-k} \\
&= \sum_{k=1}^{m+1} \binom{m}{k-1} z_1^k z_2^{m+1-k} + \sum_{k=0}^{m} \binom{m}{k} z_1^k z_2^{m+1-k} \\
&= \left[\sum_{k=1}^{m} \binom{m}{k-1} z_1^k z_2^{m+1-k} + \binom{m}{m} z_1^{m+1}\right] + \left[\binom{m}{0} z_2^{m+1} + \sum_{k=1}^{m} \binom{m}{k} z_1^k z_2^{m+1-k}\right] \\
&= \sum_{k=1}^{m} \frac{m!}{(k-1)!(m+1-k)!} z_1^k z_2^{m+1-k} + \frac{m!}{m!0!} z_1^{m+1} + \frac{m!}{0!m!} z_2^{m+1} + \sum_{k=1}^{m} \frac{m!}{k!(m-k)!} z_1^k z_2^{m+1-k} \\
&= \sum_{k=1}^{m} \left[\frac{m!}{(k-1)!(m+1-k)!} + \frac{m!}{k!(m-k)!}\right] z_1^k z_2^{m+1-k} + z_1^{m+1} + z_2^{m+1} \\
&= \sum_{k=1}^{m} \left[\frac{km!}{k(k-1)!(m+1-k)!} + \frac{(m+1-k)m!}{k!(m+1-k)(m-k)!}\right] z_1^k z_2^{m+1-k} + z_1^{m+1} + z_2^{m+1}
\end{aligned}$$

Continue the simplification.

$$(z_1+z_2)^{m+1} = \sum_{k=1}^{m}\left[\frac{km!}{k!(m+1-k)!}+\frac{(m+1-k)m!}{k!(m+1-k)!}\right]z_1^k z_2^{m+1-k} + z_1^{m+1} + z_2^{m+1}$$

$$= \sum_{k=1}^{m}\left[\frac{\cancel{km!}+(m+1)m!-\cancel{km!}}{k!(m+1-k)!}\right]z_1^k z_2^{m+1-k} + z_1^{m+1} + z_2^{m+1}$$

$$= \sum_{k=1}^{m}\frac{(m+1)!}{k!(m+1-k)!}z_1^k z_2^{m+1-k} + z_1^{m+1} + z_2^{m+1}$$

$$= \sum_{k=0}^{m}\frac{(m+1)!}{k!(m+1-k)!}z_1^k z_2^{m+1-k} + z_1^{m+1}$$

$$= \sum_{k=0}^{m+1}\frac{(m+1)!}{k!(m+1-k)!}z_1^k z_2^{m+1-k}$$

$$= \sum_{k=0}^{m+1}\binom{m+1}{k}z_1^k z_2^{m+1-k}$$

Therefore, by mathematical induction, the binomial theorem holds for complex numbers.